JOURNAL OF
PURE AND
APPLIED ALGEBRA

ELSEVIER Journal of Pure and Applied Algebra 135 (1999) 1–8

Bicommutativity for a class of graded connected Hopf algebras

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Communicated by C. Kassel; received 18 March 1997

Abstract

We show that if A is a graded connected Hopf algebra over a field of characteristic 0, such that all homogeneous elements of strictly positive degree are nilpotent, then A is commutative and cocommutative. Hence A is an exterior algebra over the primitive elements. © 1999 Elsevier Science B.V. All rights reserved.

Résumé

Nous montrons que si A est une algèbre de Hopf graduée connexe sur un corps commutatif de caractéristique nulle, où tout élément homogène de degré strictement positif est nilpotent, alors A est commutative et cocommutative, par suite A est l'algèbre extérieure sur ses éléments primitifs. © 1999 Elsevier Science B.V. All rights reserved.

1991 Math. Subj. Class.: 15A75; 16D70; 16W30; 16W50

0. Introduction

Let A be a graded connected Hopf algebra over a field of characteristic zero. All the examples we know of such algebras of finite dimension are commutative and cocommutative; thus by [6] they are exterior algebras over the primitive part. For example, if g is a semi-simple Lie algebra, then the homology $H_*(g)$ of g with trivial coefficients is known to be an exterior algebra over generators of odd degrees [3]. Other examples are given as follows. Let B be a commutative algebra (with unit) and $gl_\infty(B)$ the Lie algebra of infinite order matrices with entries in B but with only a finite number of non-zero terms. Then $H_*(gl_\infty(B))$ is isomorphic to the graded exterior algebra over $HC_{*-1}(B)$, the cyclic homology of B [5]. An analogous example is given by

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PII: S0022-4049(97)00113-8

$H_*(o_\infty(B))$ and $H_*(sp_\infty(B))$, where o_∞ and sp_∞ designate the orthogonal and symplectic Lie algebras respectively. In these cases, the cyclic homology is replaced by the dihedral homology $HD_{*-1}(B)$ of B [5]. Note that in the three last examples, we obtain connected graded Hopf algebras of infinite dimension which are also commutative and cocommutative.

It is thus natural to ask whether there exists an example of a graded connected finite-dimensional Hopf algebra which is not commutative or not cocommutative (this is the same question because of the finite dimension). We will answer this question in this paper; more precisely, we show that if every homogeneous element of strictly positive degree is nilpotent, then A is commutative and cocommutative (see Theorem 1). Hence, it is an exterior algebra. On the one hand, this gives the answer for the finite-dimensional case and on the other hand, we recover a result of Hopf [2] without any hypothesis of commutativity (see Corollary 1).

1. Results

The main result of this paper is the following theorem.

Theorem 1. *Let A be a graded, connected Hopf $\Bbbk$-algebra such that all homogeneous elements of strictly positive degree are nilpotent. Then A is commutative and cocommutative. Hence A is the exterior algebra over its primitive part.*

As a consequence, we have the following result.

Corollary 1. *Any finite-dimensional, graded, connected Hopf $\Bbbk$-algebra is commutative and cocommutative.*

The previous corollary generalizes the result of Hopf [2] who showed that if A is a finite-dimensional, commutative, graded, connected Hopf algebra, then it is an exterior algebra over generators of odd degree.

Corollary 2. *Every graded, connected Hopf algebra with a finite number of components is commutative, cocommutative and finite-dimensional.*

Remark Corollary 1 shows that the dimension of a finite-dimensional, graded, connected Hopf $\Bbbk$-algebra is of the form 2^n where n is a positive integer.

2. Notations

We denote by $\Bbbk$ a commutative field of characteristic zero, and by $(A, m, \mu, \Delta, \varepsilon, S)$ a graded, connected Hopf $\Bbbk$-algebra:

$$A = \Bbbk \oplus A_1 \oplus A_2 \oplus \cdots$$

(i) The coproduct is given by: for all $x \in A_n$,

$$\Delta(x) = x \otimes 1 + 1 \otimes x + \Delta_+(x),$$

where

$$\Delta_+(x) = \sum_{i=1}^{n-1} \left(\sum x'_i \otimes x''_{n-i} \right),$$

and $x'_i \in A_i$, $x''_{n-i} \in A_{n-i}$;

(ii) The product verifies: for all $i, j \in \mathbb{N}$

$$A_i A_j \subset A_{i+j};$$

(iii) The subspace of primitive elements of A is given by

$$P(A) = \{x \in A \mid \Delta(x) = x \otimes 1 + 1 \otimes x\};$$

(iv) The algebra A is commutative if for all homogeneous elements x, y of A,

$$xy = (-1)^{|x||y|} yx,$$

where $|x|, |y|$ denote the degree of x and y, respectively.

(v) Let $\tau : A \otimes A \to A \otimes A$ be the map defined by $\tau(x_i \otimes x_j) = (-1)^{ij} x_j \otimes x_i$, where $x_k \in A_k, k = i, j$, and $\Delta^{op} = \tau \circ \Delta$. We say that A is cocommutative if $\Delta^{op} = \Delta$.

(vi) The commutator of two homogeneous elements a, b of A is defined by

$$[a, b] = ab - (-1)^{|a||b|} ba.$$

(vii) We say that A has a finite number of components if there exists an integer m such that $A_k = \{0\}$ for all $k > m$.

3. Proof of the results

3.1. Proof of Theorem 1

We proceed in several steps. First, we investigate the subspace of primitive elements of A.

Lemma 1. *Let A be a graded connected Hopf $\Bbbk$-algebra such that all homogeneous elements of strictly positive degree are nilpotent. Then*

(1) *the non-zero homogeneous elements of $P(A)$ have odd degree;*

(2) *if a and b are homogeneous elements of $P(A)$, then $[a, b] = 0$.*

Proof. (1) Let a be a homogeneous element of $P(A)$. If $|a|$ is strictly positive and even , then $(a \otimes 1)(1 \otimes a) = (1 \otimes a)(a \otimes 1)$. Using Newton's binomial formula, we get

$$\Delta(a^n) = \Delta(a)^n = (a \otimes 1 + 1 \otimes a)^n = 1 \otimes a^n + a^n \otimes 1 + \sum_{k=1}^{n-1} \binom{n}{k} a^{n-k} \otimes a^k.$$

Since a is nilpotent, there exists a smallest integer m such that $a^m = 0$. Suppose $m \geq 2$. Then, $\Delta(a^m) = 0$ implies that

$$\sum_{k=1}^{m-1} \binom{m}{k} a^{m-k} \otimes a^k = 0.$$

Now the vectors $\{a^{m-k} \otimes a^k,\ 1 \leq k \leq m-1\}$ are linearly independent. Since $\Bbbk$ is of characteristic zero, this is impossible. Thus $m = 1$, which implies that $a = 0$.

(2) If a and b are homogeneous elements of $P(A)$, it is well known that $[a,b]$ is also primitive. Since a and b have odd degree, then $[a,b]$ has even degree; hence $[a,b] = 0$ by part (1). □

Let us denote by E the subalgebra of A generated by $P(A)$. Since the coproduct is a morphism of algebras, it is clear that E is a subcoalgebra. The fact that the antipode S is an antimorphism of algebras and that $S(x) = -x$ for each element x in $P(A)$ implies that E is a Hopf subalgebra of A. We have the following proposition.

Proposition 1. *The Hopf subalgebra E is commutative and cocommutative, hence an exterior algebra over $P(A)$.*

Proof. An element of E is a linear combination of products of primitive elements. Using part (2) of Lemma 1 and the relation

$$[ab, c] = a[b, c] + (-1)^{|b||c|}[a, c]b$$

we see that E is commutative. Since E is spanned by primitive elements, the cocommutativity is clear. So E is a commutative and cocommutative graded, connected Hopf algebra over a field of characteristic zero. We conclude by the theorem of Milnor and Moore [6]. □

The previous proposition will imply Theorem 1 once we have proved that $E = A$. Let us denote by I the augmentation ideal of E

$$I = \bigoplus_{p \geq 1} E_p, \quad \text{where } E_p = E \cap A_p.$$

Suppose that $E \neq A$, and let $L = \{c \in A \backslash E \mid \Delta_+(c) \in I \otimes I\}$.

Lemma 2. *L is not empty.*

Proof. Let c be an homogeneous element of $A \backslash E$ (assumed to be non-empty) with minimal degree n and let

$$\Delta_+(c) = \sum_{i=1}^{n-1} \left(\sum c'_i \otimes c''_{n-i} \right),$$

where $0<|c_i'|<n$, $0<|c''_{n-i}|<n$. Now, using the minimality of $|c|$ in $A\backslash E$, we deduce that c_i', c''_{n-i} are in I, and then $c\in L$. □

In the following step, we prove some facts about the set L.

Proposition 2. *Let c and t be homogeneous elements of L and $P(A)$, respectively. Then $[t,c]$ is also in $P(A)$.*

Proof. By Lemma 1, the degree of t is odd, so that, if $\Delta_+(c)=\sum_{(c)} c'\otimes c''$, then

$$\Delta([t,c])=[\Delta(t),\Delta(c)]=1\otimes[t,c]+[t,c]\otimes 1+X,$$

where $X=[1\otimes t+t\otimes 1, \sum_{(c)} c'\otimes c'']$. Let us show that $X=0$. First, we remark that if a is an homogeneous element, then $(-1)^{|a||t|}=(-1)^{|a|}$. Thus,

$$\begin{aligned}
X &= (t\otimes 1+1\otimes t)\left(\sum_{(c)} c'\otimes c''\right)-(-1)^{|c|}\left(\sum_{(c)} c'\otimes c''\right)(t\otimes 1+1\otimes t)\\
&=\sum_{(c)}(tc'\otimes c''+(-1)^{|c'|}c'\otimes tc'')-(-1)^{|c|}\sum_{(c)}((-1)^{|c''|}c't\otimes c''+c'\otimes c''t)\\
&=\sum_{(c)}(tc'\otimes c''+(-1)^{|c'|}c'\otimes tc'')-\sum_{(c)}((-1)^{|c|+|c''|}c't\otimes c''+(-1)^{|c|}c'\otimes c''t).
\end{aligned}$$

Since $(-1)^{|c|+|c''|}=(-1)^{|c|-|c''|+2|c''|}=(-1)^{|c'|}$, we have

$$\begin{aligned}
X &= \sum_{(c)}(tc'\otimes c''+(-1)^{|c'|}c'\otimes tc'')-\sum_{(c)}(-1)^{|c'|}c't\otimes c''-(-1)^{|c|}\sum_{(c)} c'\otimes c''t\\
&=\sum_{(c)}(tc'-(-1)^{|c'|}c't)\otimes c''+(-1)^{|c|}\sum_{(c)} c'\otimes((-1)^{|c'|-|c|}tc''-c''t)\\
&=\sum_{(c)}(tc'-(-1)^{|c'|}c't)\otimes c''+(-1)^{|c|}\sum_{(c)} c'\otimes((-1)^{|c''|}tc''-c''t)\\
&=\sum_{(c)}[t,c']\otimes c''-(-1)^{|c|}\sum_{(c)} c'\otimes[c'',t]=0.
\end{aligned}$$

The last equality holds because c', c'' and t are in E which is commutative by Proposition 1. □

From Lemma 1 and Proposition 2, we immediately deduce the following.

Corollary 3. *If t is primitive and c is homogeneous in L, then $[t,c]=0$ or the degree of c is even.*

Moreover, we have the following.

Proposition 3. *Any homogeneous element of L has odd degree.*

Proof. Let c be an homogeneous element of L and let C be the subalgebra of A generated by c and E. We claim that C is a Hopf subalgebra of A. Indeed, by definition of L, $\Delta(C)\subseteq C\otimes C$. Then, C is a graded connected sub-bialgebra of A; consequently, it is a Hopf subalgebra [7]. Now, Proposition 2 allows us to write every element of C as $x=\sum a_n c^n$ with a_n in E; so, if we denote by CI the left ideal of C generated by I, one easily verifies that CI is a Hopf ideal, which allows us to consider the Hopf algebra C/CI. This latter verifies the hypotheses of Lemma 1, so that every primitive element of C/CI has odd degree. Let us compute $\Delta(\bar{c})$, where $\bar{c}$ is the image of c in C/CI. We get

$$\Delta(\bar{c})=\bar{c}\otimes 1+1\otimes\bar{c}+\overline{\Delta_+(c)}.$$

Since $\Delta_+(c)\in I\otimes I$, the element $\overline{\Delta_+(c)}=0$ and $\bar{c}$ is primitive and non-zero. Indeed, if $\bar{c}=0$, then $c\in CI$. So, $c=\sum_n x_n y_n$, where $x_n\in C$ and $y_n\in I$. Since $x_n\in C$, we can write

$$x_n=\sum_k x_{n,k}c^k, \quad x_{n,k}\in E.$$

Now, each $x_{n,k}$ and y_n can be written as a sum of homogeneous elements:

$$x_{n,k}=\sum_i x_{n,k,i}, \qquad y_n=\sum_j y_{n,j}.$$

Then,

$$c=\sum_{n,i,j,k} x_{n,k,i}c^k y_{n,j},$$

Since $c\neq 0$, there exists n,i,j,k such that

$$x_{n,k,i}c^k y_{n,j}\neq 0 \quad\text{and}\quad |c|=|x_{n,k,i}c^k y_{n,j}|.$$

It follows that,

$$|c|=|x_{n,k,i}|+|c^k|+|y_{n,j}|=|x_{n,k,i}|+k|c|+|y_{n,j}|.$$

Then, $k\leq 1$. If $k=1$, then $|y_{n,j}|=0$. We get a contradiction because $y_{n,j}\in I-\{0\}$. So in each term $k=0$ and then $c\in E$, which is impossible by the choice of c. Now it is clear that $|c|=|\bar{c}|$. Hence the degree of c is odd. □

Corollary 4. *Let t be an element of $P(A)$ and c an element of L. Then,* $[t,c]=0$.

Proof. It is a trivial consequence of Proposition 3 and Corollary 3. □

In the following, we fix an homogeneous element c of L. Let $P_c=\{t_1,\dots,t_m\}$ be the set of all homogeneous primitive elements appearing in the decomposition of c_i', c_{n-i}'' as a linear combination of products of primitive elements. Let E_c be the Hopf sub-algebra of A spanned by P_c. If we denote by C_c the sub-algebra generated by E_c and c, we have the following proposition.

Proposition 4. *The subalgebra C_c is a finite-dimensional commutative Hopf subalgebra of A.*

Proof. Since $\Delta(c) \in C_c \otimes C_c$ and $\Delta(E_c) \subseteq C_c \otimes C_c$, then C_c is a sub-bialgebra of A and also a Hopf subalgebra since it is graded and connected [7]. The commutativity of E_c and $[t,c]=0$ for all t in P_c imply that C_c is commutative. Since t and c are nilpotent for all t in P_c we conclude that C_c is finite-dimensional. □

The following lemma is the key to the proof of Theorem 1.

Lemma 2. *If A is cocommutative, then Theorem 1 holds.*

Proof. Indeed, if A is cocommutative, then C_c is commutative and cocommutative; so by [6] it is spanned by primitive elements, hence $C_c \subseteq E$. Therefore, $c \in C_c \subseteq E$, which yields a contradiction. □

Now we can complete the proof of Theorem 1. Since C_c is commutative and finite-dimensional, we can consider its dual $(C_c)^*$ which is a cocommutative, graded, connected Hopf algebra. It also satisfies the nilpotency hypothesis because it is finite-dimensional. By the previous lemma, Theorem 1 holds for $(C_c)^*$ which, therefore, is commutative. This implies that C_c is cocommutative. Using again [6], the Hopf algebra C_c is generated by its primitive elements, then it is a subset of E. But $c \notin E$, hence we again get a contradiction. It follows that $E=A$ and that A is the exterior algebra over its primitive elements. □

3.2. Proof of Corollary 1

Since A is finite-dimensional, it has a finite number of components. The fact that the product is graded shows that every homogeneous element of strictly positive degree is nilpotent. Theorem 1 implies the result. □

3.3. Proof of Corollary 2

Let

$$A = \bigoplus_{k=0}^{n} A_k, \quad \text{with } A_0 = \Bbbk.$$

We can apply Theorem 1 because for all $k \in \{1,\ldots,n\}$ and all $x \in A_k$, $x^{n+1} \in A_{(n+1)k} = \{0\}$. Consequently, A is an exterior algebra over $P(A)$. If $P(A)$ were infinite-dimensional, then the exterior algebra over $P(A)$ would have an infinite number of components, hence we get a contradiction. Thus, $P(A)$ is finite-dimensional, which implies that A is finite-dimensional too. □

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